

# A new upper bound for the size of a sunflower-free family

Gábor Hegedüs

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## Abstract

We combine here Tao's slice-rank bounding method and Gröbner basis techniques and apply here to the Erdős-Rado Sunflower Conjecture.

Let  $\frac{3k}{2} \leq n \leq 3k$  be integers. We prove that if  $\mathcal{F}$  be a  $k$ -uniform family of subsets of  $[n]$  without a sunflower with 3 petals, then

$$|\mathcal{F}| \leq 3 \binom{n}{n/3}.$$

We give also some new upper bounds for the size of a sunflower-free family in  $2^{[n]}$ .

**Keywords.** sunflowers; Gröbner basis; extremal set theory

## 1 Introduction

First we introduce some notation.

Let  $[n]$  stand for the set  $\{1, 2, \dots, n\}$ . We denote the family of all subsets of  $[n]$  by  $2^{[n]}$ .

Let  $X$  be a fixed subset of  $[n]$ . For an integer  $0 \leq k \leq n$  we denote by  $\binom{X}{k}$  the family of all  $k$  element subsets of  $X$ . This is the *complete*  $k$ -uniform family.

We say that a family  $\mathcal{F}$  is *k-uniform*, if  $|F| = k$  for each  $F \in \mathcal{F}$ .

A family  $\mathcal{F} = \{F_1, \dots, F_m\}$  of subsets of  $[n]$  is a *sunflower* (or a  $\Delta$ -*system*) with  $t$  petals if

$$F_i \cap F_j = \bigcap_{s=1}^t F_s$$

for each  $1 \leq i, j \leq t$ .

Here the intersection of the members of a sunflower form its *kernel*.

Erdős and Rado conjectured the following famous statement in [8].

**Conjecture 1** *For each  $t > 2$ , there exists a constant  $C(t)$  such that if  $\mathcal{F}$  is a  $k$ -uniform set system with more than  $C(t)^k$  members, then  $\mathcal{F}$  contains a sunflower with  $t$  petals.*

Erdős offered 1000 dollars for the proof or disproof of this conjecture for  $t = 3$  (see [7]).

Erdős and Rado gave also an upper bound for the size of a  $k$ -uniform family without a sunflower with  $t$  petals in [8].

**Theorem 1.1** (*Sunflower theorem*) *If  $\mathcal{F}$  is a  $k$ -uniform set system with more than*

$$k!(t-1)^k \left(1 - \sum_{s=1}^{k-1} \frac{s}{(s+1)!(t-1)^s}\right)$$

*members, then  $\mathcal{F}$  contains a sunflower with  $t$  petals.*

Define  $F(n, t)$  to be the largest integer so that there exists a family  $\mathcal{F}$  of subsets of  $[n]$  which does not contain a sunflower with  $t$  petals and  $|\mathcal{F}| = F(n, t)$ .

Define  $\beta_t$  as

$$\beta_t := \lim_{n \rightarrow \infty} F(n, t)^{1/n}.$$

Naslund and Sawin gave the following upper bound for the size of a sunflower-free family in [13]. Their proof based on Tao's slice-rank bounding method (see the blog [14]).

**Theorem 1.2** *Let  $\mathcal{F}$  be a family of subsets of  $[n]$  without a sunflower with 3 petals. Then*

$$|\mathcal{F}| \leq 3n \left( \sum_{i=0}^{n/3} \binom{n}{i} \right).$$

Naslund and Sawin proved also the following upper bound for  $\beta_3$  in [13].

**Corollary 1.3**

$$\beta_3 \leq \frac{3}{2^{2/3}} = 1.88988\dots$$

Our main result is the following new upper bound for the size of a sunflower-free family. In the proof we mix Tao's slice-rank bounding method with Gröbner basis techniques. Our proof is a simple modification of the proof of Theorem 1 in [13].

**Theorem 1.4** *Let  $\frac{3k}{2} \leq n \leq 3k$  be integers. Let  $\mathcal{F}$  be a  $k$ -uniform family of subsets of  $[n]$  without a sunflower with 3 petals. Then*

$$|\mathcal{F}| \leq 3 \binom{n}{n/3}.$$

Theorem 1.4 implies easily the following Corollary.

**Corollary 1.5** *Let  $\mathcal{F}$  be a sunflower-free family of subsets of  $[n]$ . Then*

$$|\mathcal{F}| \leq 3 \lceil \frac{n}{3} \rceil \binom{n}{n/3} + 2 \sum_{i=0}^{\lceil n/3 \rceil} \binom{n}{i}.$$

In Section 2 we collected some useful preliminaries about the slice rang of functions and Gröbner bases. We present our proofs in Section 3.

## 2 Preliminaries

### 2.1 Slice rang

Let  $\delta$  denote in this Section the delta function.

We define first the slice rang of functions. This definition appeared first in Tao's blog [14].

Let  $A$  be a fixed finite set,  $m \geq 1$  be a fixed integer and  $\mathbb{F}$  be a field.

Recall that a function  $F : A^m \rightarrow \mathbb{F}$  has *slice-rank* one, if it has the form

$$(\mathbf{x}_1, \dots, \mathbf{x}_m) \mapsto f(\mathbf{x}_i)g(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_m),$$

for some  $i = 1, \dots, m$  and some functions  $f : A \rightarrow \mathbb{F}$ ,  $g : A^{m-1} \rightarrow \mathbb{F}$ .

The slice rank  $\text{rank}(F)$  of a function  $F : A^m \rightarrow \mathbb{F}$  is the least number of rank one functions needed to generate  $F$  as a linear combination.

For instance, if  $m = 2$ , then we get back the usual definition of the rank of a function  $F : A^2 \rightarrow \mathbb{F}$ .

Tao proved the following result about the slice rank of diagonal hypermatrices in [14] Lemma 1 (see also Lemma 4.7 in [3]).

**Theorem 2.1** *Let  $\mathbb{F}$  be a fixed field, let  $\mathcal{T} \subseteq \mathbb{F}^n$  be a finite subset and let  $c_\alpha \in \mathbb{F}$  denote a coefficient for each  $\alpha \in \mathcal{T}$ . Consider the function*

$$F(\mathbf{x}_1, \dots, \mathbf{x}_m) := \sum_{\alpha \in \mathcal{T}} c_\alpha \delta_\alpha(\mathbf{x}_1) \dots \delta_\alpha(\mathbf{x}_m) : \mathcal{T}^m \rightarrow \mathbb{F}.$$

Then

$$\text{rank}(F) = |\{\alpha \in \mathcal{T} : c_\alpha \neq 0\}|.$$

## 2.2 Gröbner theory

Let  $\mathbb{F}$  be a field. In the following  $\mathbb{F}[x_1, \dots, x_n] = \mathbb{F}[\mathbf{x}]$  denotes the ring of polynomials in commuting variables  $x_1, \dots, x_n$  over  $\mathbb{F}$ . For a subset  $F \subseteq [n]$  we write  $\mathbf{x}_F = \prod_{j \in F} x_j$ . In particular,  $\mathbf{x}_\emptyset = 1$ .

We denote by  $\mathbf{v}_F \in \{0, 1\}^n$  the characteristic vector of a set  $F \subseteq [n]$ . For a family of subsets  $\mathcal{F} \subseteq 2^{[n]}$ , define  $V(\mathcal{F})$  as the subset  $\{\mathbf{v}_F : F \in \mathcal{F}\} \subseteq \{0, 1\}^n \subseteq \mathbb{F}^n$ . A polynomial  $f \in \mathbb{F}[x_1, \dots, x_n]$  can be considered as a function from  $V(\mathcal{F})$  to  $\mathbb{F}$  in a natural way.

We can describe several interesting properties of finite set systems  $\mathcal{F} \subseteq 2^{[n]}$  as statements about *polynomial functions on  $V(\mathcal{F})$* . As for polynomial functions on  $V(\mathcal{F})$ , it is natural to consider the ideal  $I(V(\mathcal{F}))$ :

$$I(V(\mathcal{F})) := \{f \in \mathbb{F}[\mathbf{x}] : f(\mathbf{v}) = 0 \text{ whenever } \mathbf{v} \in V(\mathcal{F})\}.$$

Clearly the substitution gives an  $\mathbb{F}$  algebra homomorphism from  $\mathbb{F}[\mathbf{x}]$  to the  $\mathbb{F}$  algebra of  $\mathbb{F}$ -valued functions on  $V(\mathcal{F})$ . It is easy to verify that this homomorphism is surjective, and the kernel is exactly  $I(V(\mathcal{F}))$ . Hence we can identify the algebra  $\mathbb{F}[\mathbf{x}]/I(V(\mathcal{F}))$  and the algebra of  $\mathbb{F}$  valued functions on  $V(\mathcal{F})$ . It follows that

$$\dim_{\mathbb{F}} \mathbb{F}[\mathbf{x}]/I(V(\mathcal{F})) = |\mathcal{F}|.$$

Now we recall some basic facts about Gröbner bases and standard monomials. For details we refer to [1], [4], [5], [6].

A linear order  $\prec$  on the monomials over variables  $x_1, x_2, \dots, x_m$  is a *term order*, or *monomial order*, if 1 is the minimal element of  $\prec$ , and  $\mathbf{u}\mathbf{w} \prec \mathbf{v}\mathbf{w}$  holds for any monomials  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  with  $\mathbf{u} \prec \mathbf{v}$ . Two important term orders are the lexicographic order  $\prec_l$  and the deglex order  $\prec_d$ . We have

$$x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \prec_l x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}$$

iff  $i_k < j_k$  holds for the smallest index  $k$  such that  $i_k \neq j_k$ . The definition of the deglex order is similar: we have  $\mathbf{u} \prec_d \mathbf{v}$  iff either  $\deg \mathbf{u} < \deg \mathbf{v}$ , or  $\deg \mathbf{u} = \deg \mathbf{v}$ , and  $\mathbf{u} \prec_l \mathbf{v}$ .

The *leading monomial*  $\text{lm}(f)$  of a nonzero polynomial  $f \in \mathbb{F}[\mathbf{x}]$  is the  $\prec$ -largest monomial which appears with nonzero coefficient in the canonical form of  $f$  as a linear combination of monomials.

Let  $I$  be an ideal of  $\mathbb{F}[\mathbf{x}]$ . We say that a finite subset  $\mathcal{G} \subseteq I$  is a *Gröbner basis* of  $I$  if for every  $f \in I$  there exists a  $g \in \mathcal{G}$  such that  $\text{lm}(g)$  divides  $\text{lm}(f)$ . In other words, the leading monomials  $\text{lm}(g)$  for  $g \in \mathcal{G}$  generate the ideal of monomials  $\{\text{lm}(f) : f \in I\}$ . Consequently  $\mathcal{G}$  is actually a basis of  $I$ , i.e.  $\mathcal{G}$  generates  $I$  as an ideal of  $\mathbb{F}[\mathbf{x}]$ . A well-known fact is (cf. [5, Chapter 1, Corollary 3.12] or [1, Corollary 1.6.5, Theorem 1.9.1]) that every nonzero ideal  $I$  of  $\mathbb{F}[\mathbf{x}]$  has a Gröbner basis.

A monomial  $\mathbf{w} \in \mathbb{F}[\mathbf{x}]$  is a *standard monomial for  $I$*  if it is not a leading monomial for any  $f \in I$ . We denote by  $\text{sm}(I)$  the set of standard monomials of  $I$ .

Let  $\mathcal{F} \subseteq 2^{[n]}$  be a set family. Then the characteristic vectors in  $V(\mathcal{F})$  are all 0,1-vectors, consequently the polynomials  $x_i^2 - x_i$  all vanish on  $V(\mathcal{F})$ . It follows that the standard monomials of the ideal  $I(\mathcal{F}) := I(V(\mathcal{F}))$  are square-free monomials.

Now we give a short introduction to the notion of reduction. Let  $\mathcal{G}$  be a set of polynomials in  $\mathbb{F}[x_1, \dots, x_n]$  and let  $f \in \mathbb{F}[x_1, \dots, x_n]$  be a fixed polynomial. We can reduce  $f$  by the set  $\mathcal{G}$  with respect to  $\prec$ . This gives a new polynomial  $h \in \mathbb{F}[x_1, \dots, x_n]$ .

Here *reduction* means that we possibly repeatedly replace monomials in  $f$  by smaller ones (with respect to  $\prec$ ) in the following way: if  $w$  is a monomial occurring in  $f$  and  $\text{lm}(g)$  divides  $w$  for some  $g \in \mathcal{G}$  (i.e.  $w = \text{lm}(g)u$  for some monomial  $u$ ), then we replace  $w$  in  $f$  with  $u(\text{lm}(g) - g)$ . It is easy to verify that the monomials in  $u(\text{lm}(g) - g)$  are  $\prec$ -smaller than  $w$ .

It is a key fact that  $\text{sm}(I)$  gives a basis of the  $\mathbb{F}$ -vector-space  $\mathbb{F}[\mathbf{x}]/I$  in the sense that every polynomial  $g \in \mathbb{F}[\mathbf{x}]$  can be uniquely expressed as  $h + f$  where  $f \in I$  and  $h$  is a unique  $\mathbb{F}$ -linear combination of monomials from  $\text{sm}(I)$ . Hence if  $g \in \mathbb{F}[\mathbf{x}]$  is an arbitrary polynomial and  $\mathcal{G}$  is a Gröbner basis of  $I$ , then we can reduce  $g$  with  $\mathcal{G}$  into a linear combination of standard monomials for  $I$ . In particular,  $f \in I$  if and only if  $f$  can be  $\mathcal{G}$ -reduced to 0.

Let  $0 \leq k \leq n/2$  and denote by  $\mathcal{M}_{k,n}$  the set of all monomials  $\mathbf{x}_G$  such that  $G = \{s_1 < s_2 < \dots < s_j\} \subset [n]$  for which  $j \leq k$  and  $s_i \geq 2i$  holds for every  $i$ ,  $1 \leq i \leq j$ . These monomials  $\mathbf{x}_G$  are the *ballot monomials* of degree at most  $k$ . If  $n$  is clear from the context, then we write  $\mathcal{M}_k$  instead of the more precise  $\mathcal{M}_{k,n}$ . It is known that

$$|\mathcal{M}_k| = \binom{n}{k}.$$

In [11] we described completely the Gröbner bases and the standard monomials of the complete uniform families of all  $k$  element subsets of  $[n]$ .

**Theorem 2.2** *Let  $\prec$  an arbitrary term order such that  $x_1 \prec \dots \prec x_n$ . Let  $0 \leq k \leq n$  and  $j := \min(k, n - k)$ . Then*

$$\text{sm}(V\left(\binom{[n]}{k}\right)) = \mathcal{M}_{j,n}.$$

Let  $0 \leq k \leq n$  and  $\ell > 0$  be arbitrary integers. Define the vector system

$$\mathcal{F}(n, k, \ell) := \underbrace{V\left(\binom{[n]}{k}\right) \times \dots \times V\left(\binom{[n]}{k}\right)}_{\ell} \subseteq \{0, 1\}^{n\ell}.$$

It is easy to verify the following Corollary.

**Corollary 2.3** *Let  $\prec$  an arbitrary term order such that  $x_1 \prec \dots \prec x_n$ . Let  $0 \leq k \leq n$  and  $\ell > 0$  be arbitrary integers. Let  $j := \min(k, n - k)$ . Then*

$$\text{sm}(\mathcal{F}(n, k, \ell)) = \{x_{M_1} \cdot \dots \cdot x_{M_\ell} : x_{M_1}, \dots, x_{M_\ell} \in \mathcal{M}_{j,n}\}.$$

### 3 Proofs

**Proof of Theorem 1.4:**

Let  $\mathcal{F}$  be a  $k$ -uniform sunflower-free family of subsets of  $[n]$ .

Let  $H_1, H_2, H_3 \in \mathcal{F}$  be arbitrary subsets. Since  $\mathcal{F}$  is sunflower-free, hence if

$$\mathbf{v}(H_1) + \mathbf{v}(H_2) + \mathbf{v}(H_3) \in \{0, 1, 3\}^n,$$

then  $H_1 = H_2 = H_3$ .

Namely first suppose that  $H_1 \neq H_2$ ,  $H_1 \neq H_3$  and  $H_2 \neq H_3$ . Then the triple  $(H_1, H_2, H_3)$  is not a sunflower, hence there exist indices  $1 \leq i < j \leq 3$  such that  $(H_i \cap H_j) \setminus (H_1 \cap H_2 \cap H_3) \neq \emptyset$ . Let  $t \in (H_i \cap H_j) \setminus (H_1 \cap H_2 \cap H_3)$ . Then  $\mathbf{v}(H_1)_t + \mathbf{v}(H_2)_t + \mathbf{v}(H_3)_t = 2$ .

Suppose that  $H_1 \neq H_2$  but  $H_2 = H_3$ . Since  $|H_1| = |H_2| = k$ , hence  $H_2 \setminus H_1 \neq \emptyset$ . Let  $t \in H_2 \setminus H_1$ . Then it is easy to see that  $\mathbf{v}(H_1)_t + \mathbf{v}(H_2)_t + \mathbf{v}(H_3)_t = 2$ .

Consider the polynomial function

$$T : (\mathcal{F})^3 \rightarrow \mathbb{R}$$

given by

$$T(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \prod_{i=1}^n (2 - (x_i + y_i + z_i))$$

for each  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n), \mathbf{z} = (z_1, \dots, z_n) \in \mathcal{F} \subseteq V\binom{[n]}{k}$ .

Let  $\mathcal{G}$  denote a deglex Gröbner basis of the ideal  $I := I(\mathcal{F}(n, k, 3))$ . Let  $H$  denote the reduction of  $T$  via  $\mathcal{G}$ .

Then

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}) = T(\mathbf{x}, \mathbf{y}, \mathbf{z}) \tag{1}$$

for each  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n), \mathbf{z} = (z_1, \dots, z_n) \in \mathcal{F} \subseteq V\binom{[n]}{k}$ , because we reduced  $T$  with a Gröbner basis of the ideal  $I$ .

On the other hand

$$T(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq 0 \text{ if and only if } \mathbf{x} = \mathbf{y} = \mathbf{z} \in \mathcal{F},$$

hence by equation (1)

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq 0 \text{ if and only if } \mathbf{x} = \mathbf{y} = \mathbf{z} \in \mathcal{F}. \tag{2}$$

Let  $j := \min(k, n - k)$ .

Since  $\mathcal{F} \subseteq V\left(\binom{[n]}{k}\right)$ , hence it follows from Corollary 2.3 that we can write  $H(\mathbf{x}, \mathbf{y}, \mathbf{z})$  as a linear combination of standard monomials

$$x_I y_K z_L,$$

where  $x_I, y_K, z_L \in \mathcal{M}_{j,n}$  and  $\deg(x_I y_K z_L) \leq n$ . Here we used that  $\mathcal{G}$  is a *deglex* Gröbner basis of the ideal  $I$ .

It follows from the pigeonhole principle that at least one of  $|I|$ ,  $|K|$  and  $|L|$  is at most  $n/3$ .

First we can consider the contribution of the standard monomials to the sum for which  $|I| \leq \frac{n}{3}$ .

We can regroup this contribution as

$$\sum_M x_M g_M(\mathbf{y}, \mathbf{z}),$$

where  $M$  ranges over those subsets  $\{i_1, \dots, i_t\}$  of  $[n]$  with  $t \leq n/3$  and  $i_s \geq 2s$  for every  $1 \leq s \leq t$ . Here  $g_M : (\mathcal{F})^2 \rightarrow \mathbb{R}$  are some explicitly computable functions.

The number of such  $M$  is at most  $\binom{n}{n/3}$ , so this contribution has slice-rank at most  $\binom{n}{n/3}$ .

The remaining contributions arising from the cases  $|K| \leq \frac{n}{3}$  and  $|L| \leq \frac{n}{3}$ . But  $H$  and  $T$  are the same functions on  $\mathcal{F}(n, k, 3)$ , hence we get that

$$\text{rank}(H) = \text{rank}(T) \leq 3 \binom{n}{n/3}.$$

But it follows from Theorem 2.1 and (2) that

$$\text{rank}(H) = |\mathcal{F}|.$$

Finally we get that

$$|\mathcal{F}| \leq 3 \binom{n}{n/3}.$$

□

**Proof of Corollary 1.5:** Let  $\mathcal{F} \subseteq \{0, 1\}^n$  be a fixed sunflower-free subset. Define the families

$$\mathcal{F}(s) := \mathcal{F} \cap \binom{[n]}{s}$$



for each  $0 \leq s \leq n$ .

We have two separate cases.

1. Suppose that either  $s \geq \frac{2n}{3}$  or  $s \leq \frac{n}{3}$ . Then clearly

$$|\mathcal{F}(s)| \leq \binom{n}{s}.$$

2. Suppose that  $\frac{n}{3} \leq s \leq \frac{2n}{3}$ .

Then we can apply Theorem 1.4 for the family  $\mathcal{F}(s)$  and we get

$$|\mathcal{F}(s)| \leq 3 \binom{n}{n/3}.$$

Finally

$$|\mathcal{F}| = \sum_{s=0}^n |\mathcal{F}(s)| \leq \lceil \frac{n}{3} \rceil \left( 3 \binom{n}{n/3} \right) + 2 \sum_{i=0}^{\lceil n/3 \rceil} \binom{n}{i}.$$

□

## 4 Concluding remarks

It is easy to verify that Conjecture 1 follows immediately from the following conjecture.

**Conjecture 2** *For each  $t > 2$  there exists a constant  $C(t)$  such that if  $\mathcal{F}$  is an arbitrary  $k$ -uniform set system, which does not contain any sunflower with  $t$  petals, then  $|\cup_{F \in \mathcal{F}} F| \leq C(t)k$ .*

The following Corollary is clear.

**Corollary 4.1** *Suppose that Conjecture 2 is true for  $t = 3$ . Let  $\mathcal{F}$  be an arbitrary  $k$ -uniform set system, which does not contain any sunflower with 3 petals. Then*

$$|\mathcal{F}| \leq 3 \binom{C(t)k}{C(t)k/3}.$$

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